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THE CLOSED SOCLE OF AN ORDER

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Let Λ be an order over a domain R in a finite dimensional algebra A over the quotient field K or R, and let M be a left Λ-lattice. We generalize the work of D. Haile in [4] by associating to M an ideal $H(M)$ in R called the closed socle of M. The closed socle of M is defined as follows. A left Λ-submodule N of M is called minimal closed if $N = L \cap M$ for some minimal A-submodule L of KM. If $\mathscr{H}(M)$ denotes the sum of the minimal closed Λ-submodules of M, then $H(M) = \mathrm{Ann}_R(M/\mathscr{H}(M))$. If $M = \Lambda$ then as in [4] one has $H(\Lambda) = \mathscr{H}(\Lambda) \cap R$.

The closed socle appears to be a fairly subtle invariant and most of this paper is devoted to a study of the relationship between the structure of Λ and $H(M)$ under various hypotheses on R, Λ and M. After giving some preliminary results, we show that if Λ_1 and Λ_2 are Morita equivalent orders then $H(\Lambda_1) = H(\Lambda_2) = H(M)$ where M is any Λ_1 (Λ_2) progenerator. If Λ is a maximal order over the Dedekind domain R in a central simple algebra A over the quotient field K of R then $H(\Lambda) = R$. If R is a complete local ring and M is a projective Λ-lattice, then $H(M) = R$ if and only if M is a finite direct sum of minimal closed sublattices. Thus Λ is a direct sum of minimal closed left ideals if and only if $H(\Lambda) = R$. If G is a finite group of order $= n$ and RG is the group ring then $H(RG) = n \cdot R$. Thus, $H(RG) = R$ if and only if RG is a maximal order in KG. Generalizing 3.2 of [4], we show that if R is an integrally closed Noetherian domain and Λ is a projective maximal R-order in $M_n(K)$, then $H(\Lambda)$ is contained in the singular locus of R. We give an example of an order Λ over a discrete valuation ring R with $H(\Lambda) = R$ yet Λ is not either maximal nor a direct sum of minimal closed left ideals.

Throughout, all undefined terminology and notation will be as in [6]. H. Bass read an earlier version of this paper and made several helpful suggestions.

Section 1

Keep the notation and terminology of the introduction.

Definition 1. A left Λ-submodule N of the Λ-lattice M is called closed if it satisfies one of the following equivalent conditions.

1. For any $0 \neq r \in R$ and $m \in M$, if $rm \in N$ then $m \in N$.
2. There is an A submodule L of KM with $L \cap M = N$.

The equivalence of the definition and the next six results are straightforward generalizations of the corresponding results in [4].

Lemma 1. *There is a one-to-one order preserving correspondence between the left A-submodules L of KM and the closed left Λ-submodules N of M given by*

$$L \to L \cap M, \qquad N \to K \cdot N.$$

Since KM is a finitely generated A-module, KM satisfies the descending chain condition on submodules. Thus minimal closed submodules of M exist and are of the form $L \cap M$ for some minimal submodule L of KM.

Definition 2. Let $\mathscr{H}(M)$ denote the sum of all the minimal closed Λ-submodules of M. Let the closed socle $H(M)$ of M be $\mathrm{Ann}_R(M/\mathscr{H}(M))$.

Lemma 2. *$\mathscr{H}(\Lambda)$ is a two-sided ideal of Λ.*

Let $\mathscr{S}$ be the sum of the minimal submodules of KM. If $\mathscr{S} \neq KM$ then $\mathrm{Ann}_R(KM/\mathscr{S}) = 0$ so by an easy argument $\mathrm{Ann}_R(M/\mathscr{H}(M)) = 0$. If $\mathscr{S} = KM$ and $m_1, \ldots, m_l$ generate M over R then for each $i, m_i = \sum l_{i,j}$ where $l_{i,j} \in L_j$ with L_j a minimal Λ-submodule of KM. If we write $l_{ij} = m_{i,j}/r_{i,j}$ with $m_{i,j} \in M$ and $r_{i,j} \in R$ then $0 \neq r = \prod_{i,j} r_{i,j} \in H(M)$ so $H(M) \neq (0)$.

It follows that $H(M) \neq 0$ if and only if KM is semisimple.

Lemma 3. *Let S be a multiplicative subset of R not containing 0. Then there is a one-to-one order preserving correspondence between the closed submodules N of M and the closed submodules N' of $S^{-1}M$ given by*

$$N \to R_S N, \qquad N' \to N' \cap M.$$

Lemma 4. *Let S be a multiplicative subset of R not containing 0, then $H(S^{-1}M) = S^{-1}H(M)$.*

Proof.

$$\begin{aligned} H(S^{-1}M) &= \mathrm{Ann}_{R_S}(S^{-1}M/\mathscr{H}(S^{-1}M)) \\ &= \mathrm{Ann}_{R_S}(S^{-1}M/S^{-1}\mathscr{H}(M)) \quad \text{by Lemma 3} \\ &= \mathrm{Ann}_{R_S}(R_S \otimes M/\mathscr{H}(M)) = R_S \otimes \mathrm{Ann}_R(M/\mathscr{H}(M)) = S^{-1}H(M). \end{aligned}$$

Lemma 5. *Let p be a prime ideal in R, then $H(M_p)=H(M)_p$.*

Lemma 6. *$H(M)=\bigcap_p H(M_p)$ where p ranges over the maximal ideals of R.*

Proof. By Corollary 4, Section 3.3, Chapter II of [2] we have $H(M)=\bigcap_p H(M)_p$. From Lemma 5, $H(M)_p=H(M_p)$ and the lemma follows.

We have let $\mathscr{H}(M)$ be the sum of the minimal closed submodules of M.

Lemma 7. (a) *$\mathscr{H}(M)=\sum_{f:N\to M} f(N)$ where N runs through all Λ-modules so that $K\otimes N$ is a simple A-module and f is any Λ-homomorphism.*
(b) *If $M=M_1\oplus M_2$ then $\mathscr{H}(M)=\mathscr{H}(M_1)\oplus\mathscr{H}(M_2)$.*

Proof. Let N be a Λ-module with $K\otimes N$ simple over A and $f:N\to M$ a Λ-homomorphism. Then $1\otimes f(K\otimes N)=K\otimes f(N)$ is trivial or a simple A-submodule of $K\otimes M$ so by Lemma 1 we have $f(N)$ is trivial or a minimal closed submodule of M. Thus $\sum_{f:M\to N} f(N)=\mathscr{H}(M)$.

For part (b),

$$\mathscr{H}(M)=\sum_{f:N\to M} f(N)=\sum_{f_1+f_2:N\to M}(f_1+f_2)(N)$$

where $f_i=\pi_i f$, π_i the projection of M on M_i. Thus

$$\mathscr{H}(M)=\sum_{f_1+f_2:N\to M} f_1(N)\oplus f_2(N)=\mathscr{H}(M_1)\oplus\mathscr{H}(M_2).$$

Proposition 1. (a) *If E is a Λ-progenerator then $H(E)=H(\Lambda)$.*
(b) *If Γ is an R-order Morita equivalent to Λ then $H(\Lambda)=H(\Gamma)$.*

Proof. Let E be a Λ-progenerator, then $E\oplus E'=\Lambda^{(n)}$ for some n. By Lemma 7(b) we have $\mathscr{H}(E)\oplus\mathscr{H}(E')=\mathscr{H}(\Lambda^{(n)})=\mathscr{H}(\Lambda)^{(n)}$. If $r\in\operatorname{Ann}_R(\Lambda/\mathscr{H}(\Lambda))$ then $r\in\operatorname{Ann}_R(\Lambda^n/\mathscr{H}(\Lambda^n))$ so one can check $r\in\operatorname{Ann}_R(E/\mathscr{H}(E))$ and $H(\Lambda)\subset H(E)$. Again, since E is a progenerator, $\Lambda\oplus\Lambda'=E^{(m)}$ for some m so arguing as above $H(E)\subset H(\Lambda)$.

For part (b) let E be a Λ-lattice and $\mu:{}_\Lambda\mathscr{M}\to{}_\Gamma\mathscr{M}$ a Morita equivalence. By symmetry together with (a) it suffices to show $H(E)\subsetneq H(\mu(E))$. By Lemma 7(a),

$$\mu(\mathscr{H}(E))=\mu\left(\sum_{f:N\to E} f(N)\right)=\sum_{\mu f:\mu N\to\mu E}\mu f(\mu N).$$

It follows that $\mu(\mathscr{H}(E))=\mathscr{H}(\mu(E))$ since one can conclude from the diagram

$$\begin{array}{ccc}
{}_A\mathscr{M} & \xrightarrow{\;1\otimes\mu\;} & {}_{A_1}\mathscr{M} \\
{\scriptstyle K\otimes_R -}\uparrow & & \uparrow{\scriptstyle K\otimes_R -} \\
{}_\Lambda\mathscr{M} & \xrightarrow[\;\mu\;]{} & {}_\Gamma\mathscr{M}
\end{array}$$

that $K\otimes\mu N$ is a simple $A_1=K\otimes\Gamma$-module if and only if $K\otimes N$ is a simple A-module. Let P be a right Λ-progenerator giving the equivalence μ so $\mu(E)=P\otimes_\Lambda E$. Then

$$H(E)=\mathrm{Ann}_R(E/\mathscr{H}(E))\subseteq \mathrm{Ann}_R(P\otimes_\Lambda E/P\otimes_\Lambda\mathscr{H}(E))$$
$$=\mathrm{Ann}_R(\mu(E)/\mu(\mathscr{H}(E)))=\mathrm{Ann}_R(\mu(E)/\mathscr{H}(\mu(E))=H(\mu(E)).$$

Theorem 1. *Let Λ be an R-order in a semi-simple algebra A over the quotient field K of R. If $H(\Lambda)=R$ then Λ is a direct sum of orders in the simple factors of A.*

Proof. Let $A=A_1\oplus\cdots\oplus A_t$ be a decomposition of A into its simple factors, let $\pi_i:A\to A_i$ be the projections, let $\Lambda_i=\pi_i(\Lambda)$, and $\bar\Lambda=\Lambda_1\oplus\cdots\oplus\Lambda_t$. If L is a minimal left ideal of A, then L is an A_i-module for some i. Thus $L\cap\Lambda$ is an Λ_i-module and hence a $\bar\Lambda$-module. Thus $\mathscr{H}(\Lambda)$ is a $\bar\Lambda$ ideal contained in Λ so $\mathscr{H}(\Lambda)\subseteq\mathrm{Ann}_\Lambda(\bar\Lambda/\Lambda)$. If $H(\Lambda)=R$ then $\mathscr{H}(\Lambda)=\Lambda$ so $\Lambda=\mathrm{Ann}_\Lambda(\bar\Lambda/\Lambda)$. But $1\in\Lambda$ so $\bar\Lambda=\Lambda$.

Theorem 2. *Let Λ be a maximal order over the Dedekind domain R in the central simple algebra A over the quotient field K of R, then $H(\Lambda)=R$.*

Proof. We assume first that R is a discrete valuation ring. If $A=M_n(D)$, D a division algebra over K, let $D=\bigoplus_{i=1}^n Kx_i$, $M=\sum_{i=1}^n Rx_i$. Then $\mathcal{O}_e(M)=\{d\in D\mid dM\subseteq M\}$ is an R-order in D, contained, say, in the maximal R-order E of D. Then $M_n(E)$ is a maximal R-order in A, and by Theorem 18.7 in [6], every other order which is maximal in Λ is of the form $aM_n(E)a^{-1}$, for some $a\in A$. Thus, by Proposition 1(b), we have that $H(\Lambda)=H(aM_n(E)a^{-1})=H(M_n(E))=H(E)=R$.

Assume now that R is a Dedekind domain. Then R_p is a discrete valuation ring for every maximal ideal P or R and Λ_p is a maximal R_p-order (see [6]). Hence, as above, $H(\Lambda_p)=R_p$, and by Lemma 6 we have that $H(\Lambda)=\bigcap_p H(\Lambda_p)=\bigcap_p R_p=R$.

Theorem 10.5 of [6] implies that the conclusion of Theorem 2 remains valid if A is a direct sum of central simple K-algebras.

Theorem 3. *Let R be a complete local ring and let M be a projective lattice over the R-order Λ. Then $H(M)=R$ if and only if M is a direct sum of minimal closed submodules. Thus $H(\Lambda)=R$ if and only if Λ is a direct sum of minimal left ideals.*

Proof. If M is a direct sum of minimal closed submodules then $H(M)=R$. Conversely, suppose $H(M)=R$, then $\mathscr{H}(M)=M$. Since M is finitely generated we can find minimal closed submodules $N_1,\dots,N_t$ of M with $M=N_1+\cdots+N_t$. The natural epimorphism $f:N_1\oplus\cdots\oplus N_t\to M$ splits so $M\oplus M'\cong N_1\oplus\cdots\oplus N_t$. By the Krull–Schmidt–Remak theorem M is isomorphic to a direct sum of some subset of $\{N_i\}$.

We thank Irving Reiner for suggesting the following line of proof of the next result.

Theorem 4. *Let G be a finite group of order n and R an integrally closed Noetherian domain with quotient field K. Then $H(RG) = n \cdot R$. Thus $H(RG) = R$ if and only if RG is a maximal order in KG.*

Proof. If char $K \mid n$ then KG is not semi-simple and $H(RG) = nR = (0)$. Otherwise, A is separable K-algebra.

Let $A = \bigoplus_{i=1}^{n} Ae_i$, with $\sum_{i=1}^{n} e_i = 1$, $e_i e_j = \delta_{ij}$, where δ_{ij} is the Kronecker delta. Let $\mathscr{H}(\Lambda)$ be the sum of the minimal closed left ideals I of Λ. Then $\mathscr{H}(\Lambda) = \sum_I I \cap \Lambda$, where I ranges over all minimal left ideals of A. Since A is semi-simple, $I = Ae$ for some primitive idempotent e of A. Thus $\mathscr{H}(\Lambda) = \sum_e (Ae \cap \Lambda)$, where e ranges over all primitive idempotents of A. First we show that for every e_i we have $|G|\, e_i \in Ae_i \cap \Lambda$. If $e \in A$ is an idempotent of A, then since e is integral over R, $R[e]$ is a finitely generated R-module and subring of A. Let $M = RG$ a full R-lattice in A, then $M \cdot R[e]$ is a full R-lattice in A and so e is contained in some maximal R-order Λ_1. Now since $RG \subseteq \Lambda_1$, then $|G|\, \Lambda_1 \subseteq RG$ and so $|G| \cdot e \in RG$ (see Theorem 41.1 in [6]). Hence $|G|\, e \in (Ae \cap \Lambda)$ for every idempotent in A. $|G|\, e_i \in Ae_i \cap \Lambda$, for every i, implies that $|G| \cdot 1 = |G|\,(e_1 + \cdots + e_n) \in \mathscr{H}(\Lambda) = \sum_e Ae \cap \Lambda$ and so $|G| \cdot R \subseteq \mathscr{H}(\Lambda)$.

We now show the reverse inclusion. Let $a \in \mathscr{H}(\Lambda) \cap R$ and $e_0 = |G|^{-1} \sum_{x \in G} x$ be the central primitive idempotent in A.

Let $\bar{\Lambda} = RGe_0 \oplus \cdots \oplus RGe_n$ where the e_i are central primitive idempotents in KG. As we saw in the proof of Theorem 1, $\mathscr{H}(\Lambda)$ is a $\bar{\Lambda}$-module so $\mathscr{H}(\Lambda) = \mathscr{H}(\Lambda)e_1 \oplus \cdots \oplus \mathscr{H}(\Lambda)e_n$. Let $a \in \mathscr{H}(\Lambda)$. Then $ae_0 \in \mathscr{H}(\Lambda) \subset \Lambda$ so $a \in n \cdot RG$. Since $\mathscr{H}(RG) \subset n \cdot RG$, we have $H(RG) = \mathscr{H}(RG) \cap R \subset n \cdot R$.

Theorem 5. *Let R be a Noetherian integrally closed domain and Λ a projective maximal R-order in $M_n(K)$ where K is the quotient field of R. Then $H(\Lambda)$ is contained in the singular locus of R.*

Proof. It follows from Lemma 5 that it suffices to show $H(\Lambda_p) = R_p$ for every regular prime ideal p of R. But Λ_p is projective and maximal over R_p so by Theorem 4.3 of [1], $\Lambda_p = \mathrm{Hom}_{R_p}(E, E)$ with E a finitely generated projective R_p module. By Lemma 2 of [3], $H(\Lambda_p) = R_p$.

We conclude by giving an example of an order Λ over a discrete valuation ring R such that $H(\Lambda) = R$, yet Λ is not a direct sum of minimal closed left ideals.

Let

$$\Lambda = \left\{ \begin{pmatrix} a+5e & b+5f \\ -b+5g & a+5h \end{pmatrix} \,\middle|\, a, b, e, f, g, h \in \mathbb{Z}_5 \right\}.$$

View Λ as a $\mathbb{Z}_{(5)}$ order in $M_2(\mathbb{Q})$. The elements

$$\begin{pmatrix} 2 & -4 \\ -1 & 2 \end{pmatrix}, \quad \begin{pmatrix} 2 & 4 \\ 1 & 2 \end{pmatrix}, \quad \begin{pmatrix} 5 & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 0 & 5 \end{pmatrix}$$

generate minimal closed left ideals in Λ and

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix} - \begin{pmatrix} 2 & 4 \\ 1 & 2 \end{pmatrix} - \begin{pmatrix} 2 & -4 \\ -1 & 2 \end{pmatrix}$$

so $H(\Lambda) = R$. However, one can calculate directly that Λ has no idempotents but

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

so Λ is not a direct sum of minimal closed left ideals.

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